

$$H_\infty \neq E_\infty$$

JUSTIN NOEL

**ABSTRACT.** We provide an example of a spectrum with an  $H_\infty$  structure which does not rigidify to an  $E_3$  structure. It follows that not every  $H_\infty$  ring spectrum comes from an underlying  $E_\infty$  ring spectrum. After comparing definitions, we obtain this example by applying  $\Sigma_+^\infty$  to the counterexample to the transfer conjecture constructed by Kraines and Lada.

## 1. INTRODUCTION

In recent years there has been a renewed interest in the study of  $E_\infty$  ring spectra and their strictly commutative analogues, commutative  $S$ -algebras. Such spectra are equipped with a well-behaved theory of power operations. In the hands of an expert, this structure provides formidable computational tools which can be used to deduce a number of surprising results (for some examples, see [BMMS86, Ch. 2]).

Such operations determine and are determined by an  $H_\infty$  ring structure, the analogue of an  $E_\infty$  ring structure in the stable *homotopy* category. While this definition is formally similar to that of a space with a  $G$ -action up to homotopy, the theory of power operations is sufficiently rich that one might conjecture that every  $H_\infty$  ring spectrum is obtained by taking an  $E_\infty$  ring spectrum and then passing to the homotopy category.

We will show that such a conjecture would be a stable analogue of the transfer conjecture; the claim that the homotopy category of infinite loop spaces is equivalent to a subcategory of the homotopy category of based spaces whose objects (transfer spaces) admit certain transfer homomorphisms (see [KL79] for a more complete description).

In [KL79], Kraines and Lada construct an explicit counterexample to the transfer conjecture. In their paper, they define the notion of an  $L(n)$  space for  $0 \leq n \leq \infty$ . Under their definitions, an  $L(0)$  space is a based space, an  $L(2)$  space is a transfer space, and an  $L(\infty)$  space has a homotopy coherent action by an  $E_\infty$  operad. They also make use of the following implications

$$X \text{ is an infinite loop space} \implies X \text{ is an } E_\infty \text{ space} \implies X \text{ is an } L(\infty) \text{ space.}$$

**Theorem 1.1** ([KL79]). *Let  $s$  be a generator of  $\text{Prim}H^{30}(BU; \mathbb{Z}_{(2)})$ . Define  $KL$  by the following fibration sequence:*

$$KL \xrightarrow{i} BU_{(2)} \xrightarrow{4s} K(\mathbb{Z}_{(2)}, 30).$$

*Then  $i$  is a map of  $L(2)$  spaces, but the  $L(2)$  structure on  $KL$  does not lift to an  $E_3$  structure. In particular,  $KL$  does not admit an  $E_\infty$  structure compatible with this  $L(2)$  structure.*

After some translation we will prove the following theorem, which provides an example of an  $H_\infty$  ring spectrum whose  $H_\infty$  structure does not arise by forgetting an  $E_\infty$  structure.

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**Theorem 1.2.** *The map*

$$\Sigma_+^\infty KL \xrightarrow{\Sigma_+^\infty i} \Sigma_+^\infty BU_{(2)}$$

*is a map of  $H_\infty$  ring spectra, but the  $H_\infty$  ring structure on  $\Sigma_+^\infty KL$  does not lift to an  $E_3$  structure. In particular,  $\Sigma_+^\infty KL$  does not admit a compatible  $E_\infty$  ring structure.*

To prove this we will show that  $\Sigma_+^\infty$  takes  $L(2)$  spaces to  $H_\infty$  ring spectra and takes infinite loop spaces to  $E_\infty$  ring spectra. This follows immediately from some of the results in [May09], which we briefly recall below.

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## 2. $L(n)$ SPACES AND SPECTRA

**Notation 2.1.** *Let  $\mathcal{L}$  be the linear isometries operad. We will abuse notation and let  $L$  to denote the associated reduced monad on pointed spaces with Cartesian products, spaces under  $S^0$  with smash products, and spectra under  $S^0$  with smash products.*

In particular:

- $L$  is an endofunctor on pointed spaces satisfying

$$LY = \coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} Y^n / (\sim),$$

where  $\sim$  represents the obvious base point identifications.

- $L$  is an endofunctor on spaces under  $S^0$  satisfying

$$LY = \coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} Y^n / (\sim),$$

where  $\sim$  represents the obvious unit map identifications.

- $L$  is an endofunctor on the Lewis-May-Steinberger category of spectra (see [LMS86]) under  $S^0$  satisfying

$$LE = \bigvee_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} E^{\wedge n} / (\sim),$$

where  $\sim$  represents the obvious unit map identifications (see [EKMM97, 4.9,6.1]).

We justify this abuse of notation with the following lemma:

**Lemma 2.2** ([May09, 4.8, p. 1027]). *We have the following chain of homeomorphisms natural in based spaces<sup>1</sup>  $X$*

$$\begin{aligned} \Sigma_+^\infty LX &\equiv \Sigma^\infty (LX)_+ \\ &\cong \Sigma^\infty L(X_+) \\ &\cong L\Sigma^\infty X_+ \\ &\equiv L\Sigma_+^\infty X. \end{aligned}$$

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<sup>1</sup>It is helpful to think of this basepoint as a multiplicative unit.

For simplicity, for the remainder of this paper we will assume all spaces are non-degenerately based.

Recall that the category of  $L$ -algebras in group-like pointed spaces is equivalent to the category of infinite loop spaces. The following definition provides a categorical filtration between spaces and homotopy coherent  $L$ -algebras (which are weakly equivalent to  $L$ -algebras).

**Definition 2.3.** *A based space  $X$  is  $L(n)$  if one can construct the  $n - 1$  skeleton (in the simplicial direction) of the augmented simplicial space*

$$B(L, L, X) \xrightarrow{\mu} X,$$

*such that the canonical map*

$$X \rightarrow LX \hookrightarrow B(L, L, X)$$

*is a section of  $\mu$ .*

**Remark 2.4.** Despite the similarity in notation, we remind the reader that the *property* of being  $L(n)$  has nothing to do with the *space*  $\mathcal{L}(n)$ .

**Remark 2.5.** Note that our definition of a  $L(n)$  space is different from that of a  $Q_n$  space used in [KL79]. Kraines and Lada restrict to the case when  $X$  is connected, in which case  $L$  could be replaced with  $Q = \Omega^\infty \Sigma^\infty$ . In this respect, our definition is more general.

Our definition differs from that of Kraines and Lada in another way. Their definition of a  $Q_n$  space is a cubical analogue of the above definition, while maps of  $Q_n$  spaces are defined simplicially. Such a definition requires one to continually translate between these two worlds. We take this opportunity to propose the above alternative definition which is simpler to manipulate and can be easily adapted to any reasonable category of algebras over an operad.

Restricting to connected spaces, one can probably relate the two definitions using the Quillen equivalence between simplicial and cubical sets [Jar06, Cis06]. In any case, we only require these notions to coincide when  $X$  is connected and  $n \leq 2$ , in which case the equivalence is immediate.

We illustrate our definition with a sequence of examples (for more detailed exposition and proofs see [KL79] or [CLM76, V]).

**Example 2.6.**

- (1) By definition, every space is a  $L(0)$  space.
- (2) A space  $X$  is  $L(1)$ , if the canonical map  $X \rightarrow LX$  admits a retraction  $\mu$ .
- (3) Let  $\mu_L$  denote the structure map  $L^2 \rightarrow L$ . A space  $X$  is  $L(2)$ , if it is  $L(1)$  and we have a specified homotopy  $I \times L^2 X \rightarrow X$  between  $\mu\mu_L$  and  $\mu(\mu)$ . In other words,  $X$  is a strictly unital  $L$ -algebra in the homotopy category of pointed spaces.
- (4) A space  $X$  is  $L(\infty)$  if and only if it is a strong homotopy retract of a  $L$ -algebra. If the components of  $X$  form a group under the induced multiplication, then  $X$  is  $L(\infty)$  if and only if it has the homotopy type of an infinite loop space.

There is an obvious analogue of the above definition with based spaces replaced by spectra over  $S^0$  and  $L$  replaced by  $L$ . Since the category of  $L$ -algebras in spectra over  $S^0$  is isomorphic to the category  $E_\infty$  ring spectra [May09, 6.2], we obtain an analogous categorical filtration between spectra under  $S^0$  and  $E_\infty$  ring spectra.

Applying this equivalence to the definition of  $L(n)$  spectra, we see that the definition of an  $L(2)$  spectrum is precisely the definition of a strictly unital  $H_\infty$  ring spectrum [BMMS86].

The following proposition provides the necessary comparison to prove Theorem 1.2.

**Proposition 2.7.**

- (1) *If  $X$  is an  $L(2)$  space then  $\Sigma_+^\infty X$  is a strictly unital  $H_\infty$  ring spectrum.*
- (2) *If  $X$  is an  $L(2)$  space such that the  $H_\infty$  ring structure on  $\Sigma_+^\infty X$  rigidifies to an  $E_\infty$  structure. Then the  $L(2)$  structure on  $X$  extends to a  $L(\infty)$  structure.*

*Proof.* The first two parts are obvious from Lemma 2.2 and the comments above. For (2), the essential point is the stable maps defining an  $L$  structure on  $\Sigma_+^\infty X$  must come from the unstable  $L(2)$  structure on  $X$ , which implies we are in the image of the functor  $\Sigma_+^\infty$  from  $L$ -algebras in spaces to  $L$ -algebras in spectra in Lemma 2.2. □

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